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LETTER TO THE EDITOR

Temperley–Lieb algebra and a new integrable electronic model

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Abstract. A new correlated electron model is presented which is derived using the quantum inverse scattering method and is thus integrable in one dimension on a periodic lattice. The R-matrix is constructed from a representation of the Temperley–Lieb algebra. The associated Hamiltonian describes correlated hopping, electron pair hopping and a generalized spin interaction.

For many years the Hubbard model [1] stood as the prototype of an integrable model of correlated electrons. However, the discovery of high T_c superconductivity has promoted greater research in the area in recent years so there are now several distinct models [1–7] which are integrable in one dimension. One important method of generating such models is through the quantum inverse scattering method. In this approach, the Hamiltonian is derived from an R-matrix satisfying the Yang–Baxter equation which serves as a sufficient condition for integrability. It has been realized that the supersymmetric t–j model may be formulated in this manner [2], and subsequently other models [3, 7–9] were derived which were found to generalize the Hubbard model. A well known method of obtaining solutions of the Yang–Baxter equation is through use of the Temperley–Lieb (TL) algebra [10, 11], with the R-matrix associated with the XXZ chain being the simplest example. In this letter, a new electronic model will be derived using this approach.

The electronic model proposed in [7] was obtained through the use of a one-parameter family of representations of the Lie superalgebra $gl(2|1) \cong osp(2|2)$. In order for the Hamiltonian associated with the model to be a Hermitian operator, the free parameter is restricted to a certain range of values. However, for a particular value outside this range, the representation becomes self-dual (i.e. the representation matrices are isomorphic to their supertransposes) and may be used to construct a representation of the Birman–Wenzl–Murakami (BWM) algebra [12]. Indeed in [13], the connection between the BWM algebra and the non-Hermitian model [7] at this value of the free parameter was established. The BWM algebra contains the TL algebra as a subalgebra and this representation of the TL algebra may be used to solve the Yang–Baxter equation and construct a new integrable model, as will be demonstrated later. The important aspect of this model is that one may perform a non-unitary transformation, which preserves integrability, to recover a Hermitian Hamiltonian. This transformation has the effect of breaking the $gl(2|1)$ symmetry of the model but retains $sl(2)$ symmetry.

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Using the standard notation for the fermion operators the local Hamiltonian of the new model reads

$$\begin{aligned}H_{i,i+1} = {} & c^\dagger_{i+}c^\dagger_{i-}c_{i+1,-}c_{i+1,+} + c^\dagger_{i+1,+}c^\dagger_{i+1,-}c_{i-}c_{i+} \\ & +\varepsilon\sum_{\sigma=\pm}(c^\dagger_{i\sigma}c_{i+1\sigma} + c^\dagger_{i+1\sigma}c_{i\sigma})(n_{i,-\sigma} - n_{i+1,-\sigma})^2 - S_i^+S_{i+1}^- - S_i^-S_{i+1}^+ \\ & +(n_{i+} - n_{i+1,+})^2(n_{i-} - n_{i+1,-})^2 \end{aligned} \tag{1}$$

where $\varepsilon = \pm 1$ and $S^+ = c^\dagger_+c_-$, $S^- = c^\dagger_-c_+$. It is evident that this Hamiltonian describes electron pair hopping, correlated hopping and a generalized spin interaction. The nature of the correlated hopping terms differ from those found in [3–7] so this feature distinguishes this model from other known integrable correlated electron models. The model is also invariant under spin reflection. In the following it will be shown how this Hamiltonian can be derived through use of the TL algebra.

Let $\{|x\rangle\}_{x=1}^4$ denote an orthonormal basis for the self-dual four-dimensional $gl(2|1)$ module V. The Lie superalgebra $gl(2|1)$ has generators $\{E^i_j\}^3_{i,j=1}$ which act on this module according to

$$\begin{aligned}&E^1_2 = |2\rangle\langle 3| \qquad E^2_1 = |3\rangle\langle 2| \qquad E^1_1 = -|3\rangle\langle 3| - |4\rangle\langle 4| \qquad E^2_2 = -|2\rangle\langle 2| - |4\rangle\langle 4| \\ &E^2_3 = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| + |3\rangle\langle 4|) \qquad E^3_2 = \frac{1}{\sqrt{2}}(-|2\rangle\langle 1| + |4\rangle\langle 3|) \\ &E^1_3 = \frac{1}{\sqrt{2}}(-|1\rangle\langle 3| + |2\rangle\langle 4|) \qquad E^3_1 = \frac{1}{\sqrt{2}}(|3\rangle\langle 1| + |4\rangle\langle 2|) \\ &E^3_3 = -\tfrac{1}{2}|1\rangle\langle 1| + \tfrac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|) + \tfrac{3}{2}|4\rangle\langle 4|.\end{aligned} \tag{2}$$

The Lie superalgebra $gl(2|1)$ carries a $\mathbb{Z}_2$-grading or parity determined by

$$[E^i_j] = ([i] + [j]) \pmod 2 \tag{3}$$

where $[1] = [2] = 0$, $[3] = 1$. Consistent with this gradation, a gradation on the basis states is assigned by

$$[|1\rangle] = [|4\rangle] = 0 \qquad [|2\rangle] = [|3\rangle] = 1. \tag{4}$$

It is useful to introduce the parity operator

$$p = |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3| + |4\rangle\langle 4|$$

so the parity of a basis state is given by the eigenvalue of p.

Associated with $gl(2|1)$ there is also a co-product structure ($\mathbb{Z}_2$-graded algebra homomorphism) $\Delta : gl(2|1) \to gl(2|1) \otimes gl(2|1)$ given by

$$\Delta(E^i_j) = I \otimes E^i_j + E^i_j \otimes I \qquad \forall\ 1 \leqslant i, j \leqslant 3. \tag{5}$$

It should be noted that throughout, multiplication of all tensor products is governed by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd).$$

Under the co-product action the tensor product module $V \otimes V$ contains a one-dimensional submodule spanned by the (unnormalized) vector

$$|v\rangle = |1\rangle \otimes |4\rangle + |4\rangle \otimes |1\rangle + |3\rangle \otimes |2\rangle - |2\rangle \otimes |3\rangle.$$

Recall that the representation of $gl(2|1)$ given by (2) is not unitary but rather grade star [14]. Considered as a *right* module, $V \otimes V$ contains a submodule spanned by

$$\langle w| = \langle 1| \otimes \langle 4| + \langle 4| \otimes \langle 1| + \langle 3| \otimes \langle 2| - \langle 2| \otimes \langle 3|.$$

It then follows that the operator

$$N = |v\rangle\langle w|$$

is $gl(2|1)$ invariant with respect to the co-product action (5). This operator satisfies the relations (cf [12, 13])

$$\begin{aligned} &N^2 = 0 \\ &(I \otimes N)(N \otimes I)(I \otimes N) = (I \otimes N) \\ &(N \otimes I)(I \otimes N)(N \otimes I) = (N \otimes I) \end{aligned} \tag{6}$$

and thus gives rise to a representation of the TL algebra. Using this representation, one may construct an R-matrix and consequently a new electronic model. Essentially the local Hamiltonian of this model is given by N. However, it is obvious that N is not Hermitian and hence the model is not directly of physical interest.

Consider alternatively the operator

$$\begin{aligned} T &= |v\rangle\langle v| \\ &= N(p \otimes I) \\ &= N(I \otimes p) \end{aligned}$$

where

$$\langle v| = \langle 1| \otimes \langle 4| + \langle 4| \otimes \langle 1| + \langle 2| \otimes \langle 3| - \langle 3| \otimes \langle 2|.$$

Using the fact that $(p \otimes p)N = N$, it follows from (6) that

$$\begin{aligned} &(T \otimes I)(I \otimes T)(T \otimes I) = T \otimes I \\ &(I \otimes T)(T \otimes I)(I \otimes T) = I \otimes T. \end{aligned} \tag{7}$$

A straightforward calculation shows that

$$T^2 = 4T \tag{8}$$

indicating that T also provides a representation of the TL algebra. The operator T is not invariant with respect to the action of the $gl(2|1)$ generators but rather the even subalgebra $sl(2) \oplus u(1) \oplus u(1)$, since p commutes with the even generators and anticommutes with the odd.

Adopting the approach of [10, 11], one may construct the R-matrix

$$\check{R}(u) = I + \frac{\sinh u}{\sinh(\eta - u)} T \qquad \cosh \eta = 2$$

which satisfies the Yang–Baxter equation

$$(I \otimes \check{R}(u))(\check{R}(u+v) \otimes I)(I \otimes \check{R}(v)) = (\check{R}(v) \otimes I)(I \otimes \check{R}(u+v))(\check{R}(u) \otimes I). \tag{9}$$

Equation (9) can be verified directly using the relations (7) and (8). On the N-fold tensor product space denote

$$\check{R}(u)_{i,i+1} = I^{\otimes(i-1)} \otimes \check{R}(u) \otimes I^{(N-i-1)}$$

and define a local Hamiltonian by [15]

$$\begin{aligned} H_{i,i+1} &= \sinh \eta \frac{\mathrm{d}}{\mathrm{d}u} \check{R}(u)_{i,i+1} \bigg|_{u=0} \\ &= T_{i,i+1}. \end{aligned}$$

Finally, in view of the grading (4) identify

$$|4\rangle \equiv |0\rangle \qquad |3\rangle \equiv |+\rangle = c_+^\dagger|0\rangle$$
$$|2\rangle \equiv |-\rangle = c_-^\dagger|0\rangle \qquad |1\rangle \equiv |\pm\rangle = c_+^\dagger c_-^\dagger|0\rangle.$$

This allows $H_{i,i+1}$ to be expressed in terms of the canonical fermion operators giving the local Hamiltonian (1) with $\varepsilon = 1$. The unitary transformation

$$c_{i\sigma} \to c_{i\sigma}(1 - 2n_{i,-\sigma})$$

yields the same local Hamiltonian with $\varepsilon = -1$.

Integrability of the above model in one dimension can be seen from the following. First let P be the $\mathbb{Z}_2$-graded permutation operator defined by

$$P(|x\rangle \otimes |y\rangle) = (-1)^{[|x\rangle][|y\rangle]}|y\rangle \otimes |x\rangle \qquad \forall\ 1 \leqslant x, y \leqslant 4$$

and set $R(u) = P\check{R}(u)$. On the N-fold tensor product space one may construct the transfer matrix

$$t(u) = \mathrm{str}_0 R_{0N}(u) \ldots R_{01}(u)$$

where str_0 denotes the supertrace taken over the zeroth space. As a consequence of (9), the transfer matrices form a commuting family; viz.

$$[t(u),\ t(v)] = 0 \qquad \forall\ u, v \in \mathbb{C}$$

and so $t(u)$ may be diagonalized independently of the variable u by means of the Bethe ansatz. By imposing periodic boundary conditions, it can be shown [15] that the global Hamiltonian

$$H = \sum_{i=1}^{N-1} H_{i,i+1}$$

can alternatively be written

$$H = \sinh\eta \frac{\mathrm{d}}{\mathrm{d}u} \ln[t(u)]\bigg|_{u=0}$$

and so the diagonalization of $t(u)$ gives the diagonalization of H. Thus the model is integrable. Explicit solution of this model is currently under consideration.

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